MATH-F410 – REP. DES GROUPES & APP. A LA PHYS.

Solutions des exercices 2023-2024

Séance 8 : Algèbres de Lie (II)

1. The G_2 algebra

Section a)

By definition, the Cartan matrix has elements

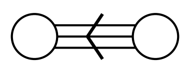
$$A_{ij} = 2\frac{\alpha_i \cdot \alpha_j}{\alpha_j^2} \ ,$$

so we get

$$A = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \ .$$

Section b)

The Dynkin diagram corresponding to the previous matrix is:



The arrow points from the longest to the shortest root, the number of lines measures $\max\{|A_{ij}|, |A_{ji}|\}$.

Section c)

For this question it is convenient to occasionally refer to the figure in the next page (this is the final result of the analysis, but it is a good idea to try to build it step by step as we describe in what follows). Also, we define positive roots for these question as those with second component positive or, if it is zero, those with first component positive (sometimes the convention starts from the first component, but this is just a matter of convention). With this choice, since simple roots have to be positive, we can set up an orthonormal basis of root space (dual vector space to the Cartan subalgebra) such that $\alpha_1 = (1, 0)$, and then $\alpha_2 = (-3/2, \sqrt{3}/2)$. This form of α_2 is obtained from the conditions $\alpha_2^2 = 3$, $\alpha_1 \cdot \alpha_2 = -3/2$, and forcing α_2 to have positive second component (so that it is positive). We now build the algebra by means of chains generated by the simple roots. Recall that, given any root β, we can form a chain generated by α_i through it

$$\beta - q\alpha_i, \beta - (q-1)\alpha_i, \ldots, \beta - \alpha_i, \beta, \beta + \alpha_i, \ldots, \beta + (p-1)\alpha_i, \beta + p\alpha_i \ ,$$

by adjoint action of $e_{\pm\alpha_i}$ on e_β, and the positive integers p and q satisfy

$$-(p - q) = 2\frac{\alpha_i \cdot \beta}{\alpha_i^2} \ .$$

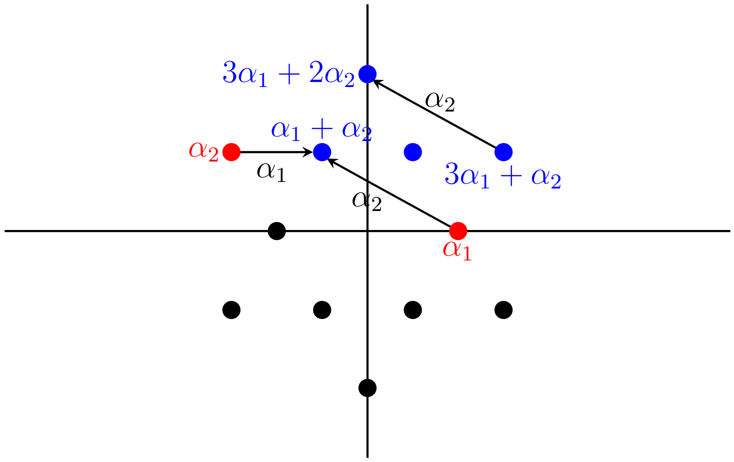


Figure 1: Root system of G_2. In red, simple roots, in blue positive (non-simple) ones, and in black negative ones. We indicate some of the steps described in the text.

The length of this chain is $p+q+1 = 2j+1$, with j the spin of the $\mathfrak{su}(2)_{\alpha_i}$ representation formed by the vectors associated with the previous roots. We assumed here α_i to be a simple root because this is enough to generate the algebra, but the previous construction is in general valid also for any root. Let us start then building chains:

- The α_1-chain through α_2 satisfies, due to the known inner product, $-(p_1 - q_1) = -3$. Furthermore, since $\alpha_2 - \alpha_1$ cannot be a root (no difference of simple roots is a root), $q_1 = 0$, so $p_1 = 3$ and we obtain as new roots $\alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2$.

- The α_2-chain through α_1 satisfies $-(p_2 - q_2) = -1$. Again, being both simple roots, $q_1 = 0$, so $p_1 = 1$ and we obtain the already known root $\alpha_1 + \alpha_2$.

- The α_2-chain through $\alpha_1 + \alpha_2$ gives nothing new due to the previous point (it is the same as the α_2-chain through α_1).

- The α_2-chain through $2\alpha_1 + \alpha_2$ cannot give anything new because neither $2\alpha_1$ nor $2(\alpha_1 + \alpha_2)$ are roots (they are multiples of roots, and these are never roots unless the multiplicative factor is -1).

- The α_2-chain through $3\alpha_1 + \alpha_2$ satisfies

$$-(p - q) = 2\frac{\alpha_2 \cdot (3\alpha_1 + \alpha_2)}{\alpha_2^2} = -1 \ .$$

  But $3\alpha_1$ cannot be a root, so $q = 0$ and $p = 1$, producing the new root $3\alpha_1 + 2\alpha_2$.

- The α_1-chain through $3\alpha_1 + 2\alpha_2$ cannot give anything new because neither $4\alpha_1 + 2\alpha_2$ nor $2(\alpha_1 + \alpha_2)$ are roots (they are multiples of existing roots).

There are no new chains we can build, so this produces all positive roots. Negative ones are just -1 times the positive roots, so these are already obtained All in all, we produce the diagram of the Figure 1.

Section d)

We have found 12 roots in total, each of them (call α a generic one) has an associated vector in the Lie algebra which diagonalizes the adjoint action of the two Cartan generators: $[h_1, e_\alpha] = \alpha(h_1)e_\alpha$ and $[h_2, e_\alpha] = \alpha(h_2)e_\alpha$. These two Cartan generators are not counted in the diagram, so we have to include them. In total, G_2 is a 14-dimensional Lie algebra.

2. Serre relations

Building the α_j-chain through α_i, we will have

$$-(p-q) = 2\frac{\alpha_i \cdot \alpha_j}{\alpha_j^2} = A_{ij} \ ,$$

where p and q determine the number of terms up and down the chain, as in the previous question. But $q = 0$ because $\alpha_i - \alpha_j$ cannot be a root for α_i, α_j simple roots. Thus, $p = -A_{ij}$. The chain of roots will be $\alpha_i, \alpha_i + \alpha_j, \ldots, \alpha_i + p\alpha_j$; for each of them there is an associated Lie algebra vector $e_{\alpha_i}, e_{\alpha_i+\alpha_j}, \ldots, e_{\alpha_i+p\alpha_j}$. These are obtained, modulo normalization, by adjoint action of the $\mathfrak{su}(2)_{\alpha_j}$ raising operator e_{α_j}, so

$$e_{\alpha_i+\alpha_j} \sim [e_{\alpha_j}, e_{\alpha_i}] = \mathrm{ad}_{e_{\alpha_j}}(e_{\alpha_i}) \ , \qquad e_{\alpha_i+2\alpha_j} \sim [e_{\alpha_j}, e_{\alpha_i+\alpha_j}] = \mathrm{ad}_{e_{\alpha_j}}(e_{\alpha_i+\alpha_j}) = \mathrm{ad}^2_{e_{\alpha_j}}(e_{\alpha_i}) \ , \ldots$$

This chain continues up to $e_{\alpha_i+p\alpha_j} = \mathrm{ad}^p_{e_{\alpha_j}}(e_{\alpha_i})$, after that the chain stops, so we conclude:

$$0 = \mathrm{ad}^{p+1}_{e_{\alpha_j}}(e_{\alpha_i}) = \mathrm{ad}^{1-A_{ij}}_{e_{\alpha_j}}(e_{\alpha_i}) \ .$$

There is a corresponding result building roots from $-\alpha_i$ with a chain generated by $-\alpha_j$. In this case, since $\alpha_i - \alpha_j$ is not a root, $-\alpha_i + \alpha_j$ is also not a root. In the $-\alpha_j$-chain through $-\alpha_i$, it must be $q = 0$ again and then $p = -A_{ij}$ as before. The conclusion in this case is:

$$0 = \mathrm{ad}^{p+1}_{e_{-\alpha_j}}(e_{-\alpha_i}) = \mathrm{ad}^{1-A_{ij}}_{e_{-\alpha_j}}(e_{-\alpha_i}) \ .$$

3. $\mathfrak{su}(N)$

This question involves heavy computations with indices. I will not provide all the details. As a warm-up, it is a good idea to check that the proposed elements of the Cartan subalgebra of $\mathfrak{su}(N)$ satisfy the requirements of being Hermitian and traceless. Take, as suggested in the question,

$$(H_m)_{ij} = \frac{1}{\sqrt{2m(m+1)}} \left[\sum_{k=1}^{m} \delta_{ik}\delta_{jk} - m\delta_{i,m+1}\delta_{j,m+1} \right] \ ,$$

for $m = 1, \ldots, N-1$. These are symmetric and real (just swap $i \leftrightarrow j$), so they are Hermitian. Furthermore, computing $\sum_{i=1}^{N} (H_m)_{ii} \sim (m - m) = 0$ (we have ignored the normalization prefactor), so they are traceless, as they should. It is a good idea to have some concrete example in mind in what follows, so we write the matrices for $\mathfrak{su}(4)$:

$$H_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \;, \quad H_2 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \;, \quad H_3 = \frac{1}{2\sqrt{6}}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} \;.$$

Section a)

Consider the matrices $(e_{ij})_{kl} = \delta_{ik}\delta_{jl}$ for $i \neq j$ (these are the matrices with a 1 in position (i,j) and the remaining entries equal to zero). Any diagonal matrix $(D)_{kl} = \sum_{i=1}^{N} d_i \delta_{ik}\delta_{il}$ satisfies:

$$([D, e_{ij}])_{kl} = \sum_{m,n=1}^{N} [d_n \delta_{nk}\delta_{nm}\delta_{im}\delta_{jl} - d_n \delta_{ik}\delta_{jm}\delta_{nm}\delta_{nl}] = (d_i - d_j)\delta_{ik}\delta_{jl} = (d_i - d_j)(e_{ij})_{kl} \;.$$

Thus, $[D, e_{ij}] = (d_i - d_j)e_{ij}$. In particular, since the H_m are diagonal, their adjoint action is diagonalized by the e_{ij}. There is a technical point here though. The e_{ij} are not Hermitian, so they are not part of the algebra of $\mathfrak{su}(N)$. Hermitian (and traceless) counterparts would be $f_{ij} = e_{ij} + e_{ji}$ and $\tilde{f}_{ij} = i(e_{ij} - e_{ji})$ for $i < j$: one can check that a basis for $\mathfrak{su}(N)$ (as a real vector space) is formed by $f_{ij}, \tilde{f}_{ij}$ (with $i < j$) and H_m. However, in this vector space, the adjoint action cannot be diagonalized. This is not so weird: after all, it happens also in $\mathfrak{su}(2)$. If we only consider Pauli matrices and their real linear combinations, we cannot diagonalize the adjoint action of σ_3. We need to consider $\sigma_1 \pm i\sigma_2$, which are (proportional to) the raising and lowering operators, to obtain $[\sigma_3, \sigma_1 \pm i\sigma_2] = \pm(\sigma_1 \pm i\sigma_2)$. Notice that $\sigma_1 \pm i\sigma_2$ are no longer Hermitian. This is exactly what we will do in $\mathfrak{su}(N)$: we work in a complexification of the Lie algebra, taking as our basis then the H_m together with $e_{ij} = \frac{1}{2}(f_{ij} - i\tilde{f}_{ij})$ for $i \neq j$. In this basis, the adjoint action of the H_m is diagonal, as we showed above.

Section b)

The weights are the eigenvalues of the elements of the Cartan subalgebra. In the fundamental representation, they are just the elements along the diagonal of H_m (in the fundamental representation the $N \times N$ matrices defining the algebra act on an N-dimensional vector space). Thus (no sum over j in this equation, we do not use Einstein's convention in this exercise to avoid confusion),

$$\nu_m^j = (H_m)_{jj} = \frac{1}{\sqrt{2m(m+1)}}\left(\sum_{k=1}^{m} \delta_{jk} - m\delta_{j,m+1}\right) \;.$$

If $j \leq m$, the sum over k contributes 1 and we get $\nu_m^j = 1/\sqrt{2m(m+1)}$. If $j = m+1$ we get a contribution from the second piece inside the parentheses, so $\nu_m^j =$

$-m/\sqrt{2m(m+1)}$. Finally, if $j > m+1$ we get a zero component of the weight. Notice that weights should be thought of as elements of the dual of the Cartan subalgebra, so they are actually r-dimensional vectors (where $r = N-1$ is the rank of the algebra) defined as

$$\nu^j = (\nu_1^j, \ldots, \nu_{N-1}^j) \ , \qquad \nu_m^j = \begin{cases} 0 & \text{if } m < j-1 \\ -\sqrt{\frac{j-1}{2j}} & \text{if } m = j-1 \\ \frac{1}{\sqrt{2m(m+1)}} & \text{if } m \geq j \end{cases} \ .$$

Here, j runs from 1 to N so we have N weights (as it corresponds to an N-dimensional representation). It is instructive to write them for $\mathfrak{su}(4)$. Each weight is a three-dimensional vector obtained by reading the elements at a fixed position in the diagonal of H_1, H_2 and H_3:

$$\nu^1 = \left(\frac{1}{2}, \frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{6}}\right) \ , \qquad \nu^2 = \left(-\frac{1}{2}, \frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{6}}\right) \ ,$$
$$\nu^3 = \left(0, -\frac{1}{\sqrt{3}}, \frac{1}{2\sqrt{6}}\right) \ , \qquad \nu^4 = \left(0, 0, -\frac{3}{2\sqrt{6}}\right) \ .$$

Section c)

As we showed in Section a),

$$\mathrm{ad}_{H_m}(e_{ij}) = [H_m, e_{ij}] = ((H_m)_{ii} - (H_m)_{jj})e_{ij} = \left(\nu_m^i - \nu_m^j\right) e_{ij} \ .$$

The eigenvalue is the root associated to e_{ij}, so $\alpha_{ij}(H_m) = \nu_m^i - \nu_m^j$. Again, roots should be though as living in the dual space to the Cartan subalgebra, this is what the notation $\alpha_{ij}(H_m)$ suggest, and in that case $\alpha_{ij} = \nu^i - \nu^j$ are the r-dimensional root vectors. Again, it is good to have the $\mathfrak{su}(4)$ example as a guide, so compute the 12 roots from the previous weights as a check!

Section d)

The second identity is actually easy,

$$\left(\sum_{j=1}^{N} \nu^j\right)_m = \sum_{j=1}^{N} \nu_m^j = \sum_{j=1}^{N} (H_m)_{jj} = \mathrm{Tr}(H_m) = 0 \ ,$$

so $\sum_{j=1}^N \nu^j = 0$. The first one is a mess, since it requires to use the general form of ν_m^j. Let us treat two cases separately. First, compute $\nu^i \cdot \nu^i$. in this case terms of the form $\sim \delta_{ik}\delta_{i,m+1}$ for $k = 1, \ldots, m$ do not contribute so

$$\nu^i \cdot \nu^i = \sum_{m=1}^{N-1} \nu_m^i \nu_m^i = \sum_{m=1}^{N-1} \frac{1}{2m(m+1)} \left[\sum_{k,k'=1}^{m} \delta_{ik}\delta_{ik'} + m^2 \delta_{i,m+1}\right]$$

$$
\begin{aligned}
&= \sum_{m=1}^{N-1} \frac{1}{2m(m+1)} \sum_{k=1}^{m} \delta_{ik} + \sum_{m=1}^{N-1} \frac{i-1}{2i} \delta_{i,m+1} \\
&= \sum_{m=i}^{N-1} \frac{1}{2m(m+1)} + \frac{i-1}{2i} = \frac{1}{2} \sum_{m=i}^{N-1} \left(\frac{1}{m} - \frac{1}{m+1} \right) + \frac{i-1}{2i} \\
&= \frac{1}{2} \left(\sum_{m=i}^{N-1} \frac{1}{m} - \sum_{m=i+1}^{N} \frac{1}{m} \right) + \frac{i-1}{2i} = \frac{1}{2i} - \frac{1}{2N} + \frac{i-1}{2i} = \frac{1}{2} - \frac{1}{2N} \ .
\end{aligned}
$$

In going from the second to the third line, notice that the $\delta_{i,m+1}$ only contributes a 1 for $i \geq 2$, but the contribution of $i = 1$ is already suppressed by the $i-1$ factor. Analogous manipulations for $i \neq j$ (we assume $i < j$ without loss of generality) produce

$$
\begin{aligned}
\nu^i \cdot \nu^j &= \sum_{m=1}^{N-1} \nu_m^i \nu_m^j = \sum_{m=1}^{N-1} \frac{1}{2m(m+1)} \left[\sum_{k=1}^{m} \delta_{ik} \sum_{k'=1}^{m} \delta_{jk'} - m\delta_{i,m+1} \sum_{k=1}^{m} \delta_{jk} - m\delta_{j,m+1} \sum_{k=1}^{m} \delta_{ik} \right] \\
&= \sum_{m=j}^{N-1} \frac{1}{2m(m+1)} - \sum_{m=1}^{N-1} \frac{1}{2(m+1)} \delta_{j,m+1} = \frac{1}{2} \sum_{m=j}^{N-1} \left(\frac{1}{m} - \frac{1}{m+1} \right) - \frac{1}{2j} = -\frac{1}{2N} \ .
\end{aligned}
$$

The non-trivial step is going to the first to the second line. Notice that the first piece within brackets is 1 if $j \leq m$ (because then immediately $i < j \leq m$), so it restricts the sum to start in $m = j$. The second piece is always zero, because if $i = m+1 < j$, then the sum over k does not contribute. Finally, the third piece contributes in $m = j-1$, because then the sum over k goes up to $j-1$ and this is $j - 1 \geq i$ (since $j > i$). All in all, we have reproduced the result in the question, $\nu^i \cdot \nu^j = -\frac{1}{2N} + \frac{1}{2}\delta_{ij}$.

Section e)

We must find first the positive roots. Notice that

$$
\alpha_{ij}(H_m) = \frac{1}{\sqrt{2m(m+1)}} \left[\sum_{k=1}^{m} (\delta_{ik} - \delta_{jk}) - m\delta_{i,m+1} + m\delta_{j,m+1} \right] \ .
$$

It is useful to split cases again. If $i < j$, the last non-vanishing component is $m = j-1$. In that case both the piece with δ_{ik} and $\delta_{j,m+1}$ contribute, so $\alpha_{ij}(H_{j-1}) = \frac{1}{\sqrt{2j(j-1)}}(1+j-1) = \sqrt{j/(2(j+1))}$ is positive. A similar analysis for $j < i$ shows that the last non-vanishing component is $\alpha_{ij}(H_{i-1}) = \frac{1}{\sqrt{2i(i-1)}}(-1-(i-1)) = -\sqrt{i/(2(i+1))}$, so negative. We conclude that positive roots are α_{ij} for $i < j$. Now, among these, which ones are simple? It is clear that

$$
\begin{aligned}
\alpha_{i(i+k)} = \nu^i - \nu^{i+k} &= (\nu^i - \nu^{i+1}) + (\nu^{i+1} - \nu^{i+1}) + \cdots + (\nu^{i+k-1} - \nu^{i+k}) \\
&= \alpha_{i(i+1)} + \alpha_{(i+1)(i+2)} + \ldots \alpha_{(i+k-1)(i+k)} \ ,
\end{aligned}
$$

for $k = 1, \ldots, N-i$. Thus, all positive roots are written as sums of roots of the form $\alpha_{i(i+1)}$ for $i = 1, \ldots, N-1$. These are the $N-1$ simple roots (there is no way to write them as sums of other two positive roots, convince yourself of this!). We will denote them with a single index as $\alpha_i \equiv \alpha_{i(i+1)}$ $(i = 1, \ldots, N-1)$.

Section f)

The Cartan matrix follows from the products previously computed:

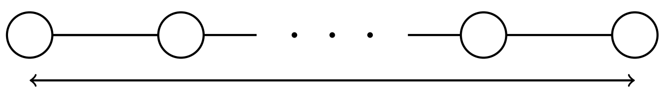
$$\alpha_i \cdot \alpha_j = (\nu^i - \nu^{i+1}) \cdot (\nu^j - \nu^{j+1}) = \frac{1}{2}\left(\delta_{i,j} - \delta_{i,j+1} - \delta_{i+1,j} + \delta_{i+1,j+1}\right) = \delta_{i,j} - \frac{1}{2}\delta_{i,j+1} - \frac{1}{2}\delta_{i,j-1} \,.$$

We see that the simple roots have unit norm and products $-1/2$ with adjacent simple roots. Thus, $A_{ij} = 2\delta_{ij} - \delta_{i,j+1} - \delta_{i,j-1}$, and the Cartan matrix is an $(N-1)\times(N-1)$ matrix of the form

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & \ldots \\ -1 & 2 & -1 & 0 & \ldots \\ 0 & -1 & 2 & -1 & \ldots \\ 0 & 0 & -1 & 2 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \,.$$

Section g)

The Dynkin diagram is just a line with $N-1$ nodes connected by single lines, because $|A_{ij}| = 1$ for $j = i \pm 1$ always:



$N-1$ nodes

4. $\mathfrak{sp}(2N)$

The construction explained in the question means we are including the previous N weights into an N-dimensional space as

$$\nu^j = \frac{1}{\sqrt{2}}\left(e^j - \frac{\sigma}{N}\right) \,, \qquad \sigma \equiv \sum_{k=1}^{N} e^k \,,$$

with $\{e^k\}$ an orthonormal basis of the auxiliary N-dimensional space. Notice that the inner products are as they should, $\nu^i \cdot \nu^j = \frac{1}{2}\delta^{ij} - \frac{1}{2N}$, and the weights are linearly dependent because $\sum_j \nu^j = 0$. We are essentially including the $N-1$ dimensional weight space (the dual to the Cartan subalgebra) into a larger, N-dimensional one, in which weights define an $(N-1)$-dimensional subspace.

Section a)

We now consider a new vector ν^{N+1} unitary and orthogonal to the ν^j, $j = 1, \ldots, N$. Expanding it in the orthonormal basis, $\nu^{N+1} = \sum_{l=1}^{N} \lambda_l e^l$, and imposing orthogonality to the ν^j we get (notice $e^l \cdot \sigma = 1$ for all $l = 1, \ldots, N$)

$$0 = \nu^j \cdot \nu^{N+1} = \frac{1}{\sqrt{2}}\lambda_j - \frac{1}{\sqrt{2}N}\sum_{l=1}^{N} \lambda_l \,.$$

Subtracting the j-th equation from the $(j+1)$-th one, we get $\lambda_{j+1} = \lambda_j$ for all $j = 1, \ldots, N-1$. All components are equal, and then they are fixed by normalization ($\nu^{N+1} \cdot \nu^{N+1} = 1$) to be $1/\sqrt{N}$. Thus,

$$\nu^{N+1} = \frac{1}{\sqrt{N}} \sum_{l=1}^{N} e^l = \frac{1}{\sqrt{N}} \sigma \ .$$

In this N-dimensional space, we are now told to consider a root system (corresponding to $\mathfrak{sp}(2N)$) given by $\alpha_i = \nu^i - \nu^{i+1}$ ($i = 1, \ldots, N-1$) plus $\alpha_N = 2\nu^N + \sqrt{2/N}\nu^{N+1}$. In terms of the canonical orthonormal basis,

$$\alpha_i = \frac{1}{\sqrt{2}} \left(e^i - e^{i+1} \right) \ , \qquad \alpha_N = \sqrt{2} e^N \ .$$

Section b)

Taking inner products one gets the different components of the Cartan matrix (in this section $i, j, \ldots$ run from 1 to $N-1$). Since the α_i are those of $\mathfrak{su}(N)$, the part of the matrix A_{ij} is just the Cartan matrix of $\mathfrak{su}(N)$. The new pieces are obtained by noting $\alpha_i \cdot \alpha_N = -\delta_{i,N-1}$, $\alpha_N^2 = 2$, so

$$A_{ij} = 2\delta_{ij} - \delta_{i,j+1} - \delta_{i,j-1} \ , \quad A_{iN} = -\delta_{i,N-1} \ , \quad A_{Ni} = -2\delta_{i,N-1} \ , \quad A_{NN} = 2 \ .$$

In matrix form,

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & \ldots & 0 & 0 \\ -1 & 2 & -1 & 0 & \ldots & 0 & 0 \\ 0 & -1 & 2 & -1 & \ldots & 0 & 0 \\ 0 & 0 & -1 & 2 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 2 & -1 \\ 0 & 0 & 0 & 0 & \ldots & -2 & 2 \end{pmatrix} \ .$$

Section c)

The Dynkin diagram is the one of $\mathfrak{su}(N)$ with an extra node appended, corresponding to the extra length 2 root, connected by a double line to the $N-1$ node (due to $\max\{|A_{N-1,N}|, |A_{N,N-1}|\} = 2$):